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ON TRIANGULATIONS OF THE CONVEX HULL OF n POINTS

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Dedicated to Paul Erdős on his seventieth birthday

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A set S of n points in Euclidean d-space determines a convex hull which can be triangulated into some number m of simplices using the points of S as vertices. We characterize those sets S for which all triangulations minimize m. This is used to characterize sets of points maximizing the volume of the smallest non-trivial simplex.

1. Introduction

Consider a set S of n points in d-dimensional Euclidean space E^d. If no hyperplane contains S we call S *d-dimensional*. We consider the d-dimensional simplex T of smallest positive volume with vertices in S. Then we consider the ratio of the volume of T to the volume of $C(S)$, the convex hull of S. We are interested in $f_d(n)$, the maximum value of this ratio for all sets S of n points. In Section 5 we show that it is exactly $(n-d)^{-1}$, and we characterize the sets S which achieve this bound. To do this we use the characterization of sets S with only 'minimal triangulations' (see sections 3 and 4 below).

For $d=2$ this is related to the Heilbronn conjecture, recently shown false by Komlós, Pintz and Szemerédi [4]. In the Heilbronn conjecture the points of S are required to be in general position (no three on a line) and the ratio under consideration is that of the area of the minimum triangle of S to the area of the smallest circle containing S. For this situation (the points in general position) the maximum value of the ratio, call it $g(n)$, is at most $O(n^{-\mu})$ for some $\mu>1$, and at least $O(n^{-2}\log n)$. (See [3], and also Problem 6.4 in section 6 below.) (If $f^*(n)$ is the maximum value of the ratio of the minimum area of a triangle of S to the area of $C(S)$ for sets S with points in general position then $f^*(n)=O(g(n))$. See [2].) Erdős, Purdy and Straus [2] considered the closely related problem of the ratio of the area of the smallest to the area of the largest triangle in S. Since the ratio of the area of the largest triangle to that of $C(S)$ is bounded below by a constant

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(see [2]), this problem also yields a ratio of order $O(1/n)$. (The exact value given in [2] is $1/[\frac{1}{2}(n-1)]$.) (For a suggestion on the higher dimensional versions of this, see Problem 6.5 below.)

We use the notion of triangulations $\mathscr{T}$ of $C(S)$, establish some of their properties, characterize certain special ones, and use them in characterizing the sets S which give the extreme values for the ratio $f_d(n)$ (Theorems 5.1 and 5.2). The characterization in Theorem 4.1 of sets S with only minimal triangulations we find interesting in itself.

Let S be a d-dimensional set of n points in E^d (S is not contained in a hyperplane). Let $C(S)$ be the convex hull of S. Then a *triangulation* of $C(S)$ is a set of nondegenerate simplices $\mathscr{T}=\{T_1, T_2, \dots, T_N\}$ with the following properties

1. $\bigcup_{\mathscr{T}} T_i = C(S)$;
2. $\text{int}(T_i)\cap\text{int}(T_j)=\emptyset$ if $i\neq j$;
3. $T_i\cap S$ is the set of vertices of T_i for each $T_i\in\mathscr{T}$.
4. $T_i\cap T_j$ is a simplex with vertices in S, for all i, j.

We are interested in the possible cardinalities of $N=|\mathscr{T}|$ for the different triangulations, especially the lower bounds. We also characterize those sets S for which all triangulations are minimal.

With each triangulation $\mathscr{T}$ of $C(S)$ we associate the *triangulation graph* $G(\mathscr{T})$ whose vertices are the simplices $T_1, \dots, T_N$ and two vertices T_i, T_j are connected by an edge if and only if $T_i\cap T_j$ is a $(d-1)$-simplex.

We are grateful to Victor Klee for pointing out the use of the triangulation graph in obtaining lower bounds for $|\mathscr{T}|$. The proof of Theorem 3.2 was supplied by him.

For $d=2$ the number of triangles in a triangulation of $C(S)$ is uniquely determined by S. In fact

$$|\mathscr{T}| = 2(n-1)-k \tag{1.1}$$

where k is the number of points of S on the boundary of $C(S)$. In particular therefor

$$|\mathscr{T}| \geqq n-2 \tag{1.2}$$

in all cases, with equality when S lies on the boundary of its convex hull.

For $d>2$ it is no longer true that S determines $|\mathscr{T}|$. However in Section 3 we investigate the generalizations of inequality (1.2) and characterize those S for which equality holds.

2. An upper bound on $|\mathscr{T}|$

The triangulation $\mathscr{T}$ is very nearly (in a sense indicated below) a simplicial d-sphere. Thus we can take advantage of the bounds known for d-sphere, in particular the 'Upper Bound Conjecture', to obtain similar bounds for $|\mathscr{T}|$. We state the relevant result below as Theorem 2.5. For a good treatment of the subject see [5].

In $(d+1)$-space consider the 'moment curve' $(x, x^2, \dots, x^{d+1})$, x a real parameter. Let $C(n+1, d+1)$ denote the convex polytope determined by any $n+1$ points on the curve ($n\geqq d+1$). That is $C(n+1, d+1)$ is the convex hull of the $n+1$ points, and has faces of dimensions $d, d-1, \dots, 1, 0$. These faces form a simplicial d-sphere, the 'boundary complex' of $C(n+1, d+1)$, denoted by $\Delta(n+1, d+1)$. For any simplicial d-sphere $\mathcal{S}$, let $f_i(\mathcal{S})$ denote the number of i-faces, $0 \leqq i \leqq d$.

For $f_i = f_i(\mathcal{S})$, $0 \leqq i \leqq d$, $f_{-1} = 1$, we define $h_i = h_i(\mathcal{S})$ by

$$h_i = \sum_{j=0}^{i} \binom{d+1-j}{d+1-i} (-1)^{i-j} f_{j-1} \quad 0 \leqq i < d+1 \tag{2.1}$$

The following relations hold ([5]):

$$f_i = \sum_{j=0}^{d+1} \binom{d+1-j}{d-i} h_j \quad 0 \leqq i \leqq d \tag{2.2}$$

$$h_i = h_{d+1-i} \quad 0 \leqq i \leqq d+1 \tag{2.3}$$

Finally, for $\mathcal{S} = \Delta(n+1, d+1)$,

$$f_i(\Delta(n+1, d+1)) = \binom{n+1}{i+1} \quad 0 \leqq i \leqq \left[\frac{1}{2}\left(d+1\right)\right] - 1. \tag{2.4}$$

The remaining values of $f_i(\Delta(n+1, d+1))$ $i \geqq [(d+1)/2]$ can be determined from (2.1)—(2.4).

Theorem 2.1 (Upper Bound Conjecture). [6] *for any simplical d-sphere $\mathcal{S}$,*

$$f_i(\mathcal{S}) \leqq f_i(\Delta(n+1, d+1)) \quad 0 \leqq i \leqq d. \quad \blacksquare$$

In order to use Theorem 2.1 we imbed $\mathcal{T}$ in a d-sphere in E^{d+1}. Let the points of S be $\bar{x}_i = (x_{il}, \ldots, x_{id}, 0)$, $1 \leqq i \leqq n$. Let S_d be the unit sphere in E^{d+1} with center at the origin. Let $\varepsilon > 0$, and form a set S' by replacing each point $\bar{x}_i$ lying in the interior of $C(S)$ by the 'lifted' point $\bar{x}_i' = (x_{il}, \ldots, x_{id}, \varepsilon)$. Let S'' be the projection from the origin of S' onto S_d. S'' has the property that the points $(x_1, \ldots, x_{d+1})$ on the 'equator', $x_{d+1} = 0$, are exactly the projections of the points of S on the boundary of $C(S)$. The other points of S'' are in the 'northern hemisphere' $x_{d+1} > 0$. S'' determines a d-sphere where the faces correspond to the simplices of $\mathcal{T}$, together with the (possibly) non-simplicial face in the equatorial plane $x_{d+1} = 0$. This face can be subdivided by adjoining to S'' the 'south pole', $(0, \ldots, 0, 1)$.

For each $(d-1)$ face on the equator (the projections of the boundary faces of $\mathcal{T}$) we adjoin $(0, \ldots, 0, 1)$ to form a d-face. The resulting complex is a simplicial d-sphere $\mathcal{T}'$.

We can now apply Theorem 2.1 to $\mathcal{T}'$ and obtain

$$|\mathcal{T}'| \leqq f_d(\Delta(+1, d+1)). \tag{2.5}$$

But from our construction $|\mathcal{T}'| = |\mathcal{T}| + l$, where l is the number of $(d-1)$-faces on the boundary of $C(S)$. Since the number is at least $d+1$, we get

$$|\mathcal{T}| = |\mathcal{T}'| - l \leqq |\mathcal{T}'| - (d+1). \tag{2.6}$$

Theorem 2.2. $| \quad | \leqq f_d(\Delta(n+1, d+1)) - (d+1)$. $\blacksquare$

Corollary 2.3. $|\mathcal{T}|=O(n^{[(d+1)/2]})$.

Proof. If $\mathcal{S}=\Delta(n+1, d+1)$ we have for $0\leqq i<[(d+1)/2]-1$ $f_i=0(n^{[(d+1)/2]})$ by (2.4). Then $h_i=O(n^{[(d+1)/2]})$ for $i\leqq[(d+1)/2]$ by (2.1) and for $i>[(d+1)/2]$ by (2.3). Then (2.2) implies $f_i=O(n^{[(d+1)/2]})$ also for $i\geqq[(d+1)/2]$. Hence in particular $f_d=O(n^{[(d+1)/2})$, and by Theorem 2.2, $|\mathcal{T}|=O(n^{[(d+1)/2]})$. ∎

As an example we consider $d=3$. By applying (2.1)—(2.4) and Theorem 2.2 we get

$$|\mathcal{T}| \leqq \frac{(n+1)(n-2)}{2}-4.$$

For $n=4, 5, 6,$ we get upper bounds respectively of $1, 5, 10$. For $n=4$ the bound is exact. For $n=5$, the maximum value of $|\mathcal{T}|$ is 4, and for $n=6$ the maximum value $|\mathcal{T}|$ can have is 8.

If we remove any vertex from a simplicial d-sphere, the resulting polytope can be thought of as a triangulation of a set in E^d (by using sterographic projection onto E^d). So if we take the maximum case, which is $\Delta(n+1, d+1)$, then this has $(n+1)(n-2)/2$ d-simplices for $d=3$ using (2.1)—(2.4). Consider a vertex v of minimum 'degree', that is a vertex meeting a minimum number of d-simplices. Since each simplex has 4 vertices, and there are $n+1$ vertices, the average degree is $\big((n+1)(n-2)/2\big)\ 4/(n+1)$. Thus v has degree no larger than that, and removing v will yield a triangulation with at least

$$\left(\frac{(n+1)(n-2)}{2}\right)\left(1-\frac{4}{n+1}\right)=\frac{(n-3)(n-2)}{2}$$

simplices. For $n=4, 5, 6,$ this gives $1, 3, 6$, respectively. These are lower bounds for the maximum of $|\mathcal{T}|$. That is, the maximum T_n of $|\mathcal{T}|$ satisfies

$$\frac{(n-3)(n-2)}{2} \leqq T_n \leqq \frac{(n+1)(n-2)}{2}-4,$$

for $d=3$.

3. Sets with minimal triangulations

We first observe that since our hypothesis states that $C(S)$ has a nonempty interior, it follows that $C(S)$ remains connected if we delete all $(d-2)$-simplices whose vertices are points of S.

Call this deleted set $C_0(S)$. Now for any triangulation $\mathcal{T}$ and any two simplices $T_i, T_j\in\mathcal{T}$ there exists a path in $C_0(S)$ which joins an interior point of T_i to an interior point of T_j. This path must correspond to a path of $G(\mathcal{T})$. We have thus proved

Lemma 3.1. *The triangulation graph $G(\mathcal{T})$ is connected.* ∎

Since $G(\mathcal{T})$ is connected we can order the vertices $T_1, T_2, \dots, T_N$ so that each subgraph induced by $\{T_1, T_2, \dots, T_m\}$ is connected, $1\leqq m\leqq N$.

Theorem 3.2. *We have*

$$|\mathscr{T}| \geqq n-d \tag{3.1}$$

with equality if and only if $G(\mathscr{T})$ is a tree.

Proof. Let $\mathscr{T}=\{T_1, T_2, \ldots, T_N\}$ be a triangulation so that the subgraph induced by $\{T_1, T_2, \ldots, T_m\}$ is connected for every $m \leqq N$. Then the set of vertices V_m which is the union of the vertices of $T_1, T_2, \ldots, T_m$ satisfies

(i) $$|V_1| = d+1$$

(ii) $$|V_{m+1}| \leqq |V_m|+1$$

since T_{m+1} has at least d vertices in V_m. Thus

$$n = |V_N| \leqq (d+1)+(N-1) = d+|\mathscr{T}|$$

which proves (3.1).

If $G(\mathscr{T})$ is a tree, then T_{m+1} is joined to exactly one T_i, $1 \leqq i \leqq m$ and therefore $|V_{m+1}|=|V_m|+1$ for all m and (3.1) becomes an equality.

If $G(\mathscr{T})$ contains a cycle and T_{m+1} is the last vertex in that cycle, then all the vertices of T_{m+1} lie in V_m and $|V_{m+1}|=|V_m|$. Thus inequality (3.1) is strict in this case. ∎

We can now characterize all trees which arise as triangulation graphs.

Theorem 3.3. *A tree τ is a triangulation graph, if and only if it has $n-d$ vertices and no vertex has valence greater than $d+1$.*

Proof. The necessity of the first condition was proved in Theorem 3.2 and the necessity of the second condition is obvious since a d-simplex has only $d+1$ faces. Now order the vertices $T_1, \ldots, T_{n-d}$ of τ so that $\{T_1, \ldots, T_m\}$ induces a connected subtree for each $m \leqq n-d$. We can now construct a set S so that $C(S)$ has a triangulation $\{T_1, \ldots, T_{n-d}\}$ with graph τ as follows:

(i)

Pick the vertices of a simplex T.

(ii)

Assume the vertices of the simplices $T_1, \ldots, T_m$ have been picked so that they all lie on the boundary of their convex hull C_m and that all faces of C_m are faces of these simplices. Now if T_{m+1} is connected to T_i, $1 \leqq i \leqq m$ then one of the faces F_i of T_i is a face of C_m and we choose a point v_{m+1} exterior to C_m but so close to the centroid of F_i that the convex hull C_{m+1} of $C_m \cup \{v_{m+1}\}$ has all the faces of C_m with the exception of F_i. We now identify T_{m+1} as the convex hull of $F_i \cup \{v_{m+1}\}$. Then the triangulation graph of C_{m+1} corresponds to the induced subgraph $\{T_1, \ldots, T_{m+1}\}$ of τ. This completes the proof by induction. ∎

Next we characterize those triangulations $\mathscr{T}$ of $C(S)$, if any, for which $G(\mathscr{T})$ is a tree.

Lemma 3.4. *The graph $G(\mathscr{T})$ of a triangulation $\mathscr{T}=\{T_1, T_2, \ldots, T_N\}$ is a tree, if and only if no $(d-2)$-face of any of the T_i intersects the interior of $C(S)$.*

Proof. Assume that $G(\mathscr{T})$ contains a cycle $\{T_1, T_2, \dots, T_l\}$. This cycle can be represented by a polygonal path P in the interior of $C(S)$ whose vertices are the centroids of the faces $T_i \cap T_{i+1}$ (where $T_{l+1} = T_1$), and whose edges are the line segments joining successive vertices. If we consider the hyperplane H determined by $T_1 \cap T_2$, then we see that P must meet both the interior of $T_1 \cap T_2$ and the exterior of $T_1 \cap T_2$ in H since P is closed. Now if P shrinks to a point it remains in the interior of $C(S)$. But at some stage it will intersect one of the boundary $(d-2)$-faces of $T_1 \cap T_2$. Thus this $(d-2)$-face meets the interior of $C(S)$.

Conversely, let F be a $(d-2)$-face of a simplex $T \in \mathscr{T}$ and let p be in the intersection of the relative interior of F with the interior of $C(S)$. Consider a circle C with center p lying in the 2-plane perpendicular to the plane of F and radius so small that it is interior to $C(S)$ and its relative interior does not intersect any $(d-2)$-face other than F. Then the simplices T_{i_j} which intersect C in cyclic order form a cycle of $G(\mathscr{T})$. ▮

Corollary 3.5. *If $C(S)$ has a triangulation with $n-d$ simplices, then S lies on the boundary of $C(S)$.*

Proof. If a point $s \in S$ lies in the interior of $C(S)$ then every triangulation of $C(S)$ has a simplex T with vertex s, and all $(d-2)$-faces of T which contain s meet the interior of $C(S)$. ▮

Corollary 3.5 is by no means a sufficient condition for the existence of a triangulation $\mathscr{T}$ for which $G(\mathscr{T})$ is a tree. For example, every simplex T which is a leaf of $G(\mathscr{T})$ must contain a vertex v of $C(S)$ so that all the incident 1-faces of $C(S)$ are 1-faces of T. In particular the number of 1-faces at v must be d. Thus for example, the regular octahedron has no tree triangulation since all vertices have 4 edges. (All the graphs are 4-cycles).

As a consequence of Lemma 3.4 we get a characterization of those S where all triangulations of $C(S)$ contain $n-d$ simplices.

Theorem 3.6. *All triangulations of $C(S)$ contain $n-d$ simplices if and only if al $(d-2)$-simplices with vertices in S lie on the boundary of $C(S)$.*

Proof. This requires only the observation that any $(d-2)$-simplex of S contains a $(d-2)$-face of one of the simplices of a triangulation of $C(S)$. We see this inductively, starting with $n=d+1$, for which there is only one simplex. For $n>d+1$ let D' be any set of $d-1$ points of S forming a $(d-2)$-simplex. $C(D')$ then contains a subset D of $d-1$ points of S forming a $(d-2)$-simplex which is minimal in the sense that $C(D) \cap S = D$ ($D=D'$ is, of course, possible). Let $x \in S-D$ be an extreme point of $C(S)$ (i.e. $x \notin C(S-\{x\})$). By induction there will be a triangulation $\mathscr{T}'$ of $C(S-\{x\})$ including D as a $(d-2)$-face of one simplex. Now we simply adjoin to $\mathscr{T}'$ all simplices exterior to $C(S-\{x\})$ which are formed by x together with a $(d-1)$-face of a simplex of $\mathscr{T}'$. ▮

As mentioned in (1.2), for $d=2$ the condition in Theorem 3.6 simply means that S lies in the boundary of $C(S)$. For $d=3$, Theorem 3.6 means that every two points of S can 'see' one another on the boundary of $C(S)$. According to a theorem of Buchman and Valentine [1] this means that $C(S)$ is either a cone whose base is a convex $(n-1)$-gon (not necessarily strictly convex) or a convex triangular prism (in the projective sense; that is, the triangular bases are not necessarily parallel, and the lateral edges lie on parallel lines or on concurrent lines) and $n-6$ points (other than the 6 corners) lie on the lateral edges of the prism. (There is a degenerate case of the prism where one of the bases is a single point.)

4. Characterization of sets S for which $G(\mathscr{T})$ is a tree for every triangulation of $C(S)$, $d \geqq 3$

From Theorems 3.2 and 3.6 we know that the equivalent property that S must satisfy so that all triangulations are trees is:

$F(d)$: *S is a d-dimensional set with all of its $(d-2)$ simplices on the boundary of $C(S)$.*

Then the generalization from the $d=3$ case mentioned above is:

Theorem 4.1. *For $d \geqq 3$ S satisfies $F(d)$ if and only if S has one of the two following structures:*

(A_d): *$C(S)$ is a 'prism' in the projective sense: $U=\{u_1, \dots, u_d\}$, $V=\{v_1, \dots, v_d\}$ are two 'bases' and $C(S)$ is the convex hull of $U \cup V$, where the lines $u_i v_i$ are either all parallel or all concurrent at a single point. In this case all the points in $S-(U \cup V)$ lie on the 'ribs' of the prism. (The degenerate case of $|U|=1$ or $|V|=1$ is possible.)*
$(B_d:)$: *$C(S)$ is a 'cone': all points of S except one lie in a hyperplane H, and for $d>3$ $H \cap S$ satisfies A_{d-1} or B_{d-1}, and for $d=3$ $C(H \cap S)$ is a convex polygon with $H \cap S$ on its boundary.*

Corollary 4.2. *A necessary and sufficient condition that every triangulation $\mathscr{T}$ of $C(S)$ be a tree is that S satisfies A_d or B_d.*

Proof of Theorem 4.1. The proof will be by induction on d. For $d=3$ the result is due to Buchman and Valentine [1], as mentioned above. Thus we assume $d>3$ and that the theorem holds for all $d'<d$. The sufficiency of conditions A_d and B_d is easy to see. First assume A_d holds. Then any $(d-2)$-simplex of S (i.e., formed by $d-1$ points of S) lies on at most $d-1$ of the ribs $u_i v_i$ of $C(S)$, and is thus on a boundary hyperplane. If B_d holds, then any $(d-2)$-simplex of S either lies in the 'base' $H \cap S$, and is thus on the boundary of $C(S)$, or includes the 'apex' p and meets the base in a $(d-3)$-simplex. But induction (applied to $H \cap S$ and using $F(d-1)$; see Lemma 4.4 below) implies that this $(d-3)$-simplex is on the (relative) boundary of the base, and thus, when p is adjoined, the resulting $(d-2)$-simplex is on the boundary of $C(S)$.

Before proving the necessity of A_d or B_d we require a few lemmas.

Lemma 4.3. *Let $k \geqq 3$, and suppose K is a set with $|K| \geqq k+2$ and satisfying $F(k)$. Then some $k+1$ points of k lie on a common hyperplane.*

Proof. Let $P_1, \dots, P_{k+1}$ determine a k-simplex D. Then the $k+1$ face-hyperplanes of D divide E^k into $2^{k+1}-1$ regions. Let Q be another point of K. If Q lies on any of the face-hyperplanes, then K has $k+1$ points in that hyperplane, and the conclusion of the lemma holds. We can assume then that Q is in the interior of one of the regions. In particular D itself is such a region. If Q is in D, then any $(k-2)$-simplex containing Q is not on the boundary, violating $F(k)$. Thus Q is in the interior of one of the other regions R.

R can be described in a nice way. Let $I \subseteq \{1, \dots, k+1\}$ be the set of indices such that for $i \in I$, the face-hyperplane not containing P_i separates P_i and R. Let F_I denote the face of D determined by the P_i, $i \in I$, and let G_I denote the face determined by the P_i, $i \notin I$. Then (by linear algebra) for each point $r \in R$ there are unique points $f \in F_I$ and $g \in G_I$ so that the line extending from f through g meets r. If r is the interior point Q of R, then f is interior to F_I or $|I|=1$, and similarly g is interior to G_I or $|I|=k$. In any case, the line fg contains interior points of D, and hence g is on the interior of $C(K)$.

Now both the faces determined by G_I and by $F_I \cup \{Q\}$ contain g, and one of $|G_I|$ and $|F_I \cup \{Q\}|$ is no larger than $[k/2]+1$. Thus any $(k-2)$-face containing it will contain g and thus violate $F(k)$. Hence Q is not an interior point of R, and the lemma is proved. ∎

We note that we only used for this lemma what we might call
$F'(k)$: *S is a k-dimensional set with all its $[k/2]$-dimensional simplices on the boundary of $C(S)$.*

Now let S be a k-dimensional set. Let H be a hyperplane determined by k of the points of S.

Lemma 4.4. *If S satisfies $F(k)$, then $H \cap S$ satisfies $F(k-1)$ in H, $k \geqq 3$.*

Proof. Suppose $k-2$ points of $S \cap H$ determine a $(k-3)$-simplex D containing a point interior to $C(S \cap H)$, say q. Let p be a point of $S-(S \cap H)$. Then the line pq must have points interior to $C(S)$, contradicting the assumption that $C(D \cup \{p\})$ must be on the boundary of $C(S)$. ∎

Lemma 4.5. *Consider the following cases:*

(i) S is a k-dimensional set, $k \geqq 4$, and there is a $(k-2)$-space H_{k-2} such that $S-(H_{k-2} \cap S) \supseteq \{a, b, c\}$, where a, b, c are not collinear, and the three lines they determine are all skew to H_{k-2}. Moreover, there is a set $Q \subseteq H_{k-2} \cap S$ disjoint from the plane of a, b, c such that Q is the vertex set of a $(k-2)$-simplex.

(ii) S is a 5-dimensional set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$, where each of the sets $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ are 2-dimensional and form strictly convex quadrilaterals.

(iii) S is a 4-dimensional set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3\}$ where a_1, a_2, a_3, a_4 are coplanar and form a strictly convex quadrilateral, and the points b_1, b_2, b_3 are collinear.

Then in cases (i), (ii), (iii) respectively S does not satisfy $F(k)$, $F(5)$, $F(4)$.

Proof. In case (i) assume $F(k)$. We derive a contradiction. Applying Lemmas 4.3 and 4.4 repeatedly to $Q \cup \{a, b, c\}$, we must eventually obtain a 2-dimensional subset F of four points (satisfying $F(2)$). F must contain all three of a, b, c for otherwise it would be a subset of one of the sets $Q \cup \{a, b\}$, $Q \cup \{a, c\}$, $Q \cup \{b, c\}$

all of which are simplices, since ab, ac, bc are skew to H_{k-2}. So $F=\{a, b, c, q\}$, $q \in Q$. This contradicts the assumption that Q was disjoint from the plane abc.

In case (ii) we may take the points to be:

$$a_1 = (00000) \qquad b_1 = (00100)$$

$$a_2 = (00001) \qquad b_2 = (01000)$$

$$a_3 = (00010) \qquad b_3 = (10100)$$

$$a_4 = (00011) \qquad b_4 = (11000)$$

Then the 3-face determined by a_2, a_3, b_2, b_3 contains the point (1/4, 1/4, 1/4. 1/4, 1/4). an interior point of $C(S)$. Thus $F(5)$ is not satisfied.

In case (iii) we may take the points to be:

$$a_1 = (0001) \qquad b_1 = (1000)$$

$$a_2 = (0010) \qquad b_1 = (0100)$$

$$a_3 = (0011) \qquad b_3 = (x, 1-x, 0, 0)$$

$$a_4 = (0000)$$

Then the 2-plane determined by a_1, a_2, b_3 contains the point $(x/3, (1-x)/3, 1/3, 1/3)$, an interior point of $C(S)$, violating $F(4)$.

We now proceed to prove the necessity of conditions A_d and B_d. Recall that $d \geqq 4$. Assume that S satisfies $F(d)$, and let H be a hyperplane so that $H \cap S$ has the maximum possible number of points of S. By Lemma 4.4 $H \cap S$ satisfies $F(d-1)$ in H. There are two possibilities:

(a) $H \cap S$ satisfies A_{d-1}
(b) $H \cap S$ satisfies B_{d-1}.

In either case, if $S-(S \cap H)$ has only one point, then S satisfies B_d; and the conclusion of the theorem is satisfied. So we can assume $S-(S \cap H)$ contains at least two points x and y.

Consider (b) first. Let p be the apex of the cone $H \cap S$, and let H_{d-2} be the $(d-2)$-space containing the base $B=H_{d-2} \cap S$. $B \cup \{x, y\}$ must be d-dimensional, or any hyperplane containing $B \cup \{x, y\}$ would violate the maximality of H. B must be $(d-2)$-dimensional (not less) for the same reason. Thus xy is skew to H_{d-2}. Similarly px and py are ksew to H_{d-2}. Either p, x, y are collinear, or they determine a plane P.

Assume first pxy is a plane P. Then by Lemma 4.5 (b) this leads to a contradiction unless the plane P also contains points of B, and $B-P$ is contained in some $(d-3)$-space H_{d-3}. $B \cap P$ actually contains exactly one point, q. For if it contains two points, then $P \cap H_{d-2}$ contains a line L. Not all of xy, yp, xp can avoid L, and thus must meet H_{d-2}. This would violate the fact that these lines are skew to H_{d-2}. So P contains x, y, p, q. Further, $xypq$ forms a strictly convex

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quadrilateral. To see this, apply Lemmas 4.3 and 4.4 repeatedly to the set $\{x, y, p, q\} \cup B'$, where $B' \cup \{q\}$ is a $(d-2)$-simplex in B. We ultimately get a 2-dimensional set F satisfying $F(2)$. Since $B' \cup \{q, x, y\}$, $B' \cup \{q, x, p\}$, $B' \cup \{q, p, y\}$ are simplices, the only possibility for the set F is $\{x, y, p, q\}$. So F is convex, by $F(2)$, and F is strictly convex since xy, xp, yp are skew to B, and x, y, p are assumed not to be collinear. Now in H_{d-3} suppose there are only $d-2$ points of S. Then if $r \in H_{d-3}$, there is a hyperplane determined by $[(S \cap H_{d-3}) - \{r\} \cup \{x, y, p, q\}$, which violates the maximality of H. Hence H_{d-3} contains at least $(d-1)$ points of S. Then by repeated application of Lemmas 4.3 and 4.4, if $d \geqq 5$, then H_{d-3} must contain a coplanar set of 4 points, satisfying $F(2)$, and hence convex, say a_1, a_2, a_3, a_4. But if this set is strictly convex, Lemma 4.5 (ii) applies to $\{x, y, p, q, a_1, a_2, a_3, a_4\}$, which then does not satisfy $F(5)$. This leads to a contradiction from repeated application of Lemma 4.4 to S. On the other hand, if a_1, a_2, a_3, a_4 is not strictly convex, then three are collinear, say a_1, a_2, a_3. But now Lemma 4.5 (iii) applies to $\{x, y, p, q, a_1, a_2, a_3\}$, leading to the contradiction that this set does not satisfy $F(4)$, as in the previous case. There remains the case $d=4$. Here H_{d-3} is a line with at least $(4-1)=3$ points, and once again Lemma 4.5 (iii) applies and leads to a contradiction. This completes the case where p, x, y determine a plane.

Now assume pxy is a line, which is skew to H_{d-2}. By induction and Lemma 4.4, B is either a cone or a prism. If it is a cone with apex p', then $(B - \{p'\}) \cup \cup \{x, y, p\}$ is a $(d-1)$-dimensional set, violating maximality of H. So B is a prism and not a cone. Then any of its 2-dimensional faces (determined by two of its parallel (or concurrent) ribs) has four points forming a strictly convex quadrilateral. This quadrilateral, together with x, y, p must satisfy $F(4)$, by repeated use of Lemma 4.4. But this contradicts Lemma 4.5 (iii). This completes the case where pxy is a line, and thus case (b), providing a contradiction for every case with $S - (S \cap H) \supseteqq \{x, y\}$.

Now consider case (a): $H \cap S$ satisfies A_{d-1}. Let $U = \{u_1, \dots, u_{d-1}\}$ be one of the base simplices of the prism $H \cap S$. If $H \cap S - \{u_i\}$ is $(d-2)$-dimensional, then case (b) above is applicable. Thus we may assume the prism has two disjoint simplicial bases, U and $V = \{v_1, \dots, v_{d-1}\}$, with possibly, but not necessarily, one more point at the common intersection of $u_i v_i$ (if it exists).

Now we may assume that x and y are chosen so that the line xy is skew to at least one of the laterial faces of the prism. For if not, then all possible choices of x and y are on a line including a point common to all faces of the prism. But then all points of $S - (S \cap H)$ must be collinear and together with $S \cap H$ form a d-dimensional prism, satisfying A_d. Let the face F skew to xy be determined by $\{u_2, \dots, u_{d-1}\} \cup \{v_2, \dots, v_{d-1}\}$. At least one of the two triples x, y, u_1 or x, y, v_2 will not be collinear, since xy is skew to F, say x, y, u_1. Now xu_1 and yu_1 are also skew to F, since F and u_1 are in H whereas x and y are not. Then the $(d-2)$-space containing F, together with x, y and u_1 satisfy Lemma 4.5 (i). This implies S does not satisfy $F(d)$, a contradiction. This completes the proof of Theorem 4.1. ∎

5. Maximizing the volume of the minimal simplex determined by n points S with given volume of $C(S)$

As before we consider n-tuples S of points in E^d not all on a hyperplane and now define

$$f(S) = \min_T \operatorname{vol}(T)/\operatorname{vol}(C(S))$$

where T ranges all over nondegenerate simplices with vertices in S. Set $f_d(n) = \sup_P f(S)$. From Theorem 3.2 we get

Theorem 5.1. $f_d(n) = 1/(n-d)$

Proof. The inequality $f_d(n) \leqq 1/(n-d)$ is an immediate consequence of Theorem 3.2. To see that the upper bound is attained let S be the vertices of a simplex, T, plus $n-d-1$ equally spaced points on one of the edges E of T. Here $C(S)$ has exactly one triangulation $\mathcal{T} = \{T_1, \dots, T_{n-d}\}$ where each T_i contains the $(d-2)$-face F opposite to E in T and two consecutive vertices on E. ▮

We can actually characterize all d-dimensional sets S of n points so that $f(S) = f_d(n)$. Let S be such a set, and let $\mathcal{T}$ be any triangulation of $C(S)$. By Lemma 3.2 and Theorem 5.1, $\mathcal{T}$ has exactly $n-d$ simplices, all equal in volume. It is easy to see that for $d=1$ S must consist of n points equally spaced on a line, and for $d=2$ either S consists of points equally spaced on two parallel lines, with the spacing the same on both lines, or else S is the set S_6 of six points formed by the vertices of a triangle together with the midpoints of the edges. This can be generalized to arbitrary d as follows:

Theorem 5.2. *Let S be a d-dimensional set of n points. Then $f(S) = f_d(n)$ if and only if one of the following conditions holds.*

(A'_d): *$C(S)$ is a prism with parallel ribs (the bases are not necessarily parallel) and S divides these ribs all into equal length segments. That is, if $S' = \{u_i, \dots, u_d\}$, $S'' = \{v_i, \dots, v_d\}$ are the two bases, then $C(S)$ is the convex hull of $S' \cup S''$. The ribs $u_i v_i$ are all parallel. Let l_i be the length of $u_i v_i$, and m_i the number of points of S on $u_i v_i$. Then $l_i/(m_i - 1) = l_j/(m_j - 1)$ for all i, j with $m_i, m_j > 1$, and the m_i points are equally spaced on $u_i v_i$. We allow the degenerate cases where $m_i = 1$, $l_i = 0$, $u_i = v_i$, and $u_i v_i$ is considered parallel to any line.*

(B'_d): *$C(S)$ is a cone, where S consists of S_6 together with $d-2$ other points.*

Proof. For $d = 1, 2$ we already saw that A'_d or B'_d holds, and they are clearly sufficient for $f(S) = f_d(n)$. Since for any $\mathcal{T}$ we saw that there are $n-d$ simplices, all of equal volume, we can invoke Lemma 3.6 and Theorem 4.1 to conclude for $d \geqq 3$ that S satisfies A_d or B_d. First consider the case A_d. It is easy to see that in this case the ribs of the prism must be parallel, and all are subdivided by S into the same length equal segments. For if not there will be two 'basic' d-simplices of different volumes, where a d-simplex T of S is 'basic' if its $d+1$ vertices are the only points of S it contains.

Since any basic simplex is part of some triangulation $\mathcal{T}$ (by the same argument as in Theorem 3.6), this would violate the observations about $\mathcal{T}$ above. Thus A'_d is satisfied.

On the other hand, suppose B_d holds. We prove by induction on d that A'_d or B'_d holds. For $d \geqq 3$ we have a cone with apex p and a base hyperplane H so that $S-(S \cap H)=\{p\}$. Then every basic $(d-1)$-simplex of $S \cap H$ must have the same volume, or by adjoining p we would obtain basic d-simplices of different volumes. Then $H \cap S$ satisfies A'_{d-1} or B'_{d-1} by induction. If A'_{d-1} is satisfied by $H \cap S$, then S satisfies A'_d, where p is a degenerate rib of the prism. If $H \cap S$ satisfies B'_{d-1}, then S satisfies B'_d.

Conversely, it is easy to see that A'_d and B'_d imply that all basic simplices T have the same volume. This gives $f(S)=f_d(n)$ and completes the proof. ▮

6. Further Problems

6.1 *Characterize triangulation graphs which are not trees.*

6.2 *Characterize the different sets of triangulations that arise from a single set S.* In the case where all triangulations are trees, there are only two possibilities by Theorem 4.1. First, if S consists of some points determining an l-dimensional prism, together with $d-l$ points in general position, then only one tree occurs, which is the path of $n-d$ points. On the other hand, the only other alternative is that S consists of $n-d+2$ points on a convex polygon together with $d-2$ points in general position. By Theorem 3.3 then the trees arising in this case all have $n-d$ vertices and vertex degrees at most 3. It may be that all such trees are in the set, or only some subset of them depending on how many of the $n-d+2$ points are actually vertices of the polygon, for instance.

6.3 *Characterize sets with a unique triangulation graph.* Besides the examples mentioned in the previous problem, there is the example of the vertices of the regular octahedron, which determine only the 4-cycle C_4 as a triangulation graph.

6.4 *Determine how close to general position the points of S may be and still have* $f(S)=O(n^{-1})$. For $d=2$, we know that if we don't allow three points colinear, then the ratio for the Heilbronn problem is $O(n^{-\mu})$, $\mu>1$. If we allow $\sqrt{n}$ points on a line, then considering the $\sqrt{n} \times \sqrt{n}$ square lattice we see that $f(S)=O(n^{-1})$. We conjecture that if we allow no more than $n^{1/2-\varepsilon}$ points to be colinear, then $f(S)=o(n^{-1})$. In general, we suspect that $n^{((d-1)/d)}$ is the corresponding critical number for points on a hyperplane in d-dimensions.

6.5 *Generalize the results of* [2] *on the ratio of the areas of maximal and minimal triangles to d-dimensions.* This is given as problem 1 [2]: Let S be a set of n points not all on a hyperplane in E^d. Let $g(S)$ denote the ratio of the minimal to the maximal volume of nondegenerate simplices with vertices in S. Find $g_d(n)=\sup_S g(S)$. The conjecture, by analogy, is that

$$g_d(n) = \frac{1}{[(n-1)/d]}.$$

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